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The Elastic-Viscoelastic Correspondence Principle for Functionally Graded Materials, Revisited

Paulino and Jin [Paulino, G. H., and Jin, Z.-H., 2001, "Correspondence Principle in Viscoelastic Functionally Graded Materials," ASME J. Appl. Mech., ***68****, pp. 129–132], have recently shown that the viscoelastic correspondence principle remains valid for a linearly isotropic viscoelastic functionally graded material with separable relaxation (or creep) functions in space and time. This paper revisits this issue by addressing some subtle points regarding this result and examines the reasons behind the success or failure of the correspondence principle for viscoelastic functionally graded materials. For the inseparable class of nonhomogeneous materials, the correspondence principle fails because of an inconsistency between the replacements of the moduli and of their derivatives. A simple but informative one-dimensional example, involving an exponentially graded material, is used to further clarify these reasons.* [DOI: 10.1115/1.1533805]

1 Introduction

The present study is motivated by a recent investigation of Paulino and Jin [1] on the correspondence principle in functionally graded materials (FGMs), as discussed below. Such materials are those in which the composition and volume fraction of the constituents vary gradually, giving a nonuniform microstructure with continuously graded macroproperties. Various thermomechanical problems related to FGMs have been studied, for example, constitutive modeling, [2], thermal stresses, [3], fracture behavior, [4], viscoelastic fracture, [5–7], time-dependent stress analysis, [8], strain gradient effects, [9], plate bending, [10], higher order theory, [11], and so on. Comprehensive reviews on several aspects of FGMs may be found in the article by Hirai [12], the chapter by Paulino et al. [13], and the book by Suresh and Mortensen [14].

One of the primary application areas of FGMs is high-temperature technology. For example, in a ceramic/metal FGM, the ceramic offers thermal barrier effects and protects the metal from corrosion and oxidation while the FGM is toughened and strengthened by the metallic composition. Materials will exhibit creep and stress relaxation behavior at high temperatures. Viscoelasticity offers a reasonable basis for the study of phenomenological behavior of creep and stress relaxation. The correspondence principle is probably the most useful tool in viscoelasticity because the Laplace transform of the viscoelastic solution can be directly obtained from the existing elastic solution. The viscoelastic correspondence principle, unfortunately, does not hold, in general, for FGMs. Paulino and Jin [1], however, have proved that the correspondence principle of viscoelasticity and thermoviscoelasticity is valid for a class of FGMs where the relaxation functions in shear and dilatation, $\mu(\mathbf{x},t)$ and $K(\mathbf{x},t)$, have separable forms, i.e., $\mu(\mathbf{x},t)=\mu(\mathbf{x})g(t)$ and $K(\mathbf{x},t)=K(\mathbf{x})f(t)$, respectively, in which $\mathbf{x}$ denotes Cartesian coordinates, t is time, and $f(t)$ and $g(t)$ are admissible, but otherwise arbitrary functions of time. For convenience of presentation, let this class of viscoelastic materials be called the "separable class." Thus the rest of the materials constitute the so called "inseparable class." Paulino and Jin have applied the correspondence principle to this "separable class" of FGMs to study crack problems under antiplane shear, [5,6], and in-plane loading, [7]. Other authors studying crack problems in nonhomogeneous viscoelastic materials have directly solved the governing viscoelastic equations without using the correspondence principle. For example, Schovanec et al. have considered stationary cracks, [15], quasi-static crack propagation, [16], and dynamic crack propagation, [17], in nonhomogeneous viscoelastic media under antiplane shear conditions. Schovanec and Walton have also considered quasi-static propagation of a plane-strain mode I crack in a power-law inhomogeneous linearly viscoelastic body, [18], and calculated the corresponding energy release rate, [19]. Although a "separable class" of viscoelastic materials were studied in Refs. [15] to [19], no use of the correspondence principle was made in their work. As a result, the mathematical calculations in these papers become quite complicated and involved.

It is important to mention some older work related to the subject of this paper. Hilton and Clementes [20] and Hashin [21] have considered viscoelastic problems with piecewise constant properties. Their problems are not directly relevant to the case of continuously varying elastic moduli under consideration in the present work. Schapery [22] has, in fact, considered the continuously varying case in which the (spatially variable) elastic moduli also depend on the Laplace transform parameter s. The present work is concerned only with the *usual class of nonhomogeneous elastic materials* in which the moduli are functions only of the spatial coordinates $\mathbf{x}$, not of time or of the Laplace parameter.

The present paper supplements that by Paulino and Jin [1]. It is first shown that the success or failure of the correspondence principle for linear nonhomogeneous viscoelastic materials rests upon the forms of the spatial derivatives of the relaxation functions, since these quantities appear in the equilibrium equations. This discussion is followed by a simple but informative one-dimensional example for which closed-form solutions are obtained for a Maxwell material under tensile loading with (a) a separable and (b) an inseparable relaxation function. Two kinds of boundary conditions, displacement prescribed and mixed, are considered for this example.

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2 The Viscoelastic Correspondence Principle for Functionally Graded Materials

Some of the governing equations for nonhomogeneous isotropic linearly elastic and viscoelastic materials, under quasi-static deformation, in the physical and Laplace transformed domains, are outlined below. The standard equations for homogeneous viscoelastic materials are available in many references, e.g., Christensen [23].

2.1 Elasticity. The well-known constitutive equation for linear elastic behavior is

$$\sigma_{ij}(\mathbf{x},t)=\lambda(\mathbf{x})\varepsilon_{kk}(\mathbf{x},t)\delta_{ij}+2\mu(\mathbf{x})\varepsilon_{ij}(\mathbf{x},t) \tag{1}$$

where σ_{ij} and ε_{ij} are components of the stress and strain tensors, respectively, λ and μ are Lamé parameters and δ_{ij} are components of the Kronecker delta. It is useful to note that $\lambda=K-(2/3)\mu$ where K and μ are the bulk and shear moduli, respectively, of the material.

Taking Laplace transforms (when they exist) defined as $\overline{f}(s)=\int_0^\infty f(t)\exp^{-st}dt$, (1) becomes

$$\overline{\sigma}_{ij}(\mathbf{x},s)=\lambda(\mathbf{x})\overline{\varepsilon}_{kk}(\mathbf{x},s)\delta_{ij}+2\mu(\mathbf{x})\overline{\varepsilon}_{ij}(\mathbf{x},s). \tag{2}$$

Applying the equilibrium equation (in the absence of body forces) in the Laplace transform domain to (2), one obtains

$$\begin{aligned}0=\overline{\sigma}_{ij,j}(\mathbf{x},s)&=\lambda(\mathbf{x})\overline{\varepsilon}_{kk,i}(\mathbf{x},s)+2\mu(\mathbf{x})\overline{\varepsilon}_{ij,j}(\mathbf{x},s)+\lambda_{,i}(\mathbf{x})\overline{\varepsilon}_{kk}(\mathbf{x},s)\\&\quad+2\mu_{,j}(\mathbf{x})\overline{\varepsilon}_{ij}(\mathbf{x},s)\end{aligned} \tag{3}$$

where $(\cdot)_{,j}=\partial(\cdot)/\partial x_j$.

2.2 Viscoelasticity. This time, the integral form of the constitutive equation, with relaxation functions $\lambda(\mathbf{x},t)$ and $\mu(\mathbf{x},t)$, is

$$\begin{aligned}\sigma_{ij}(\mathbf{x},t)=&\int_0^t\lambda(\mathbf{x},t-\tau)\frac{\partial\varepsilon_{kk}}{\partial\tau}(\mathbf{x},\tau)\delta_{ij}d\tau\\&+2\int_0^t\mu(\mathbf{x},t-\tau)\frac{\partial\varepsilon_{ij}}{\partial\tau}(\mathbf{x},\tau)d\tau\end{aligned} \tag{4}$$

and its Laplace transform is

$$\overline{\sigma}_{ij}(\mathbf{x},s)=s\overline{\lambda}(\mathbf{x},s)\overline{\varepsilon}_{kk}(\mathbf{x},s)\delta_{ij}+2s\overline{\mu}(\mathbf{x},s)\overline{\varepsilon}_{ij}(\mathbf{x},s). \tag{5}$$

Applying the equilibrium equation to (5) results in

$$\begin{aligned}0=\overline{\sigma}_{ij,j}(\mathbf{x},s)&=s\overline{\lambda}(\mathbf{x},s)\overline{\varepsilon}_{kk,i}(\mathbf{x},s)+2s\overline{\mu}(\mathbf{x},s)\overline{\varepsilon}_{ij,j}(\mathbf{x},s)\\&\quad+s\overline{\lambda}_{,i}(\mathbf{x},s)\overline{\varepsilon}_{kk}(\mathbf{x},s)+2s\overline{\mu}_{,j}(\mathbf{x},s)\overline{\varepsilon}_{ij}(\mathbf{x},s).\end{aligned} \tag{6}$$

2.3 Range of Validity of the Correspondence Principle *Consider a nonhomogeneous isotropic linear elastic material with shear and bulk moduli $\mu(\mathbf{x})$ and $K(\mathbf{x})$, respectively. Now consider a boundary value problem for a body B with a fixed boundary ∂B composed of this material. Let ∂B_u and ∂B_τ ($\partial B=\partial B_u\cup\partial B_\tau$) be parts of the boundary on which the displacements and tractions, respectively, are prescribed. It is also assumed that ∂B_u and ∂B_τ do not vary in time. The applied boundary displacements and/or tractions are allowed to be (slowly varying) functions of time–therefore, the fields in B—displacement, strain and stress, can also be functions of time. Inertia and body forces are neglected here. In this situation, the usual (quasi-static) viscoelastic correspondence principle remains valid in general in the separable case, i.e., when the (viscoelastic) relaxation functions in shear and in dilatation have the forms $\mu(\mathbf{x},t)=\mu(\mathbf{x})g(t)$, $K(\mathbf{x},t)=K(\mathbf{x})f(t)$, respectively, where f(t) and g(t) are sufficiently well behaved but otherwise arbitrary functions of time. For the inseparable case, the viscoelastic correspondence principle is not valid in general.*

2.4 Success of Correspondence Principle for the "Separable Class". The crucial step is a comparison of Eqs. (3) and (6) and the replacements:

$$\lambda(\mathbf{x})\Rightarrow s\overline{\lambda}(\mathbf{x},s),\quad \mu(\mathbf{x})\Rightarrow s\overline{\mu}(\mathbf{x},s) \tag{7}$$

$$\lambda_{,i}(\mathbf{x})\Rightarrow s\overline{\lambda}_{,i}(\mathbf{x},s),\quad \mu_{,i}(\mathbf{x})\Rightarrow s\overline{\mu}_{,i}(\mathbf{x},s). \tag{8}$$

A sufficient condition for the validity of the correspondence principle is fulfilled by the "separable class" of linear viscoelastic materials where

$$\lambda(\mathbf{x},t)=\lambda(\mathbf{x})h(t),\quad \mu(\mathbf{x},t)=\mu(\mathbf{x})g(t). \tag{9}$$

Now

$$\overline{\lambda}(\mathbf{x},s)=\lambda(\mathbf{x})\overline{h}(s),\quad \overline{\mu}(\mathbf{x},s)=\mu(\mathbf{x})\overline{g}(s) \tag{10}$$

so that

$$\overline{\lambda}_{,i}(\mathbf{x},s)=\lambda_{,i}(\mathbf{x})\overline{h}(s),\quad \overline{\mu}_{,i}(\mathbf{x},s)=\mu_{,i}(\mathbf{x})\overline{g}(s). \tag{11}$$

Therefore, for this "separable class" of materials, Eq. (6) becomes

$$\begin{aligned}0=&s\lambda(\mathbf{x})\overline{h}(s)\overline{\varepsilon}_{kk,i}(\mathbf{x},s)+2s\mu(\mathbf{x})\overline{g}(s)\overline{\varepsilon}_{ij,j}(\mathbf{x},s)\\&+s\lambda_{,i}(\mathbf{x})\overline{h}(s)\overline{\varepsilon}_{kk}(\mathbf{x},s)+2s\mu_{,j}(\mathbf{x})\overline{g}(s)\overline{\varepsilon}_{ij}(\mathbf{x},s).\end{aligned} \tag{12}$$

With the replacements (7) and (10) for the relaxation functions, and (8) and (11) for their derivatives, Eqs. (3) and (12) are compatible; therefore, the correspondence principle is valid for this "separable class" of viscoelastic materials.

2.5 Failure of the Correspondence Principle for the "Inseparable Class". It is now observed that the replacements (7), which work for homogeneous problems, do not, in general, work in the inseparable case. The reason for this is that the replacements (8) are, in general, inconsistent, in the sense that the spatial dependence of $\overline{\lambda}_{,i}(\mathbf{x},s)$ and (or) $\overline{\mu}_{,i}(\mathbf{x},s)$ can be quite different from those (that) of $\lambda_{,i}(\mathbf{x})$ and (or) $\mu_{,i}(\mathbf{x})$, respectively. This issue is rather subtle and the failure of the correspondence principle for the inseparable case is demonstrated by means of a simple example in Section 3 of this paper.

3 An Illustrative One-dimensional Example

This section presents a simple one-dimensional example (see Fig. 1), considering exponentially graded properties, to illustrate the various issues regarding the validity or not of the correspondence principle for viscoelastic functionally graded materials (FGMs). Materials with exponential gradation have been widely used in the technical literature—see, for example, Refs. [13,14]. In the present example, closed-form solutions are obtained for a nonhomogeneous Maxwell material under tensile loading with (a) a separable and (b) an inseparable relaxation function. Two types of boundary conditions, displacement prescribed and mixed, are considered here.

3.1 Relaxation Function in Tension. Consider a nonhomogeneous Maxwell material with tensile parameters $E(x)$ and $\eta(x)$ as shown in Fig. 1(*a*). The relaxation function of this material in tension, together with its Laplace transform, are, [23],

$$E(x,t)=E(x)\exp\left[\frac{-E(x)t}{\eta(x)}\right],\quad \overline{E}(x,s)=\frac{E(x)}{s+E(x)/\eta(x)}. \tag{13}$$

Two cases are considered next:
(a) separable: $E(x)=E_0e^{-\alpha x}$, $\eta(x)=\eta_0e^{-\alpha x}$.
(b) inseparable: $E(x)=E_0e^{-\alpha x}$, $\eta(x)=\eta_0$.
In the above, E_0, η_0, and α are material constants. Notice that α has units $[\text{length}]^{-1}$ and thus $1/\alpha$ expresses the length scale of inhomogeneity. Such an additional length scale characterizes an FGM and influences its material behavior.

3.2 Range of Validity of Correspondence Principle

Separable Class. For case (a), which belongs to the "separable class," one has

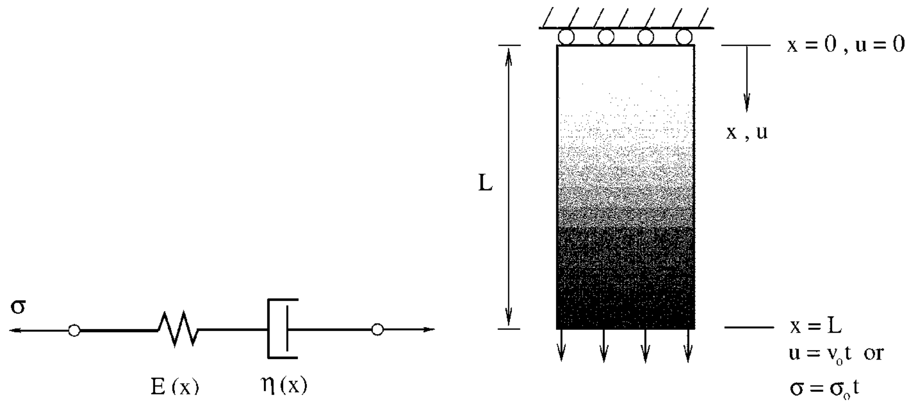

Fig. 1 One-dimensional example; (*a*) nonhomogeneous Maxwell material; (*b*) bar under tensile loading

$$E(x)\Rightarrow s\bar{E}(x,s)=\frac{sE(x)}{s+E_0/\eta_0} \tag{14}$$

$$E'(x)\Rightarrow s\frac{\partial}{\partial x}\bar{E}(x,s)=\frac{sE'(x)}{s+E_0/\eta_0}. \tag{15}$$

In this case, the replacements for $E(x)$ and $E'(x)$ are consistent (see Eqs. (3) and (12)) and the correspondence principle remains valid.

Inseparable class. Now

$$E(x)\Rightarrow s\bar{E}(x,s)=\frac{sE(x)}{s+E(x)/\eta_0}=\frac{sE_0e^{-\alpha x}}{s+(E_0/\eta_0)e^{-\alpha x}}. \tag{16}$$

A consistent replacement for $E'(x)$ should be

$$\frac{sE'(x)}{s+E'(x)/\eta_0}=\frac{-\alpha sE_0e^{-\alpha x}}{s-\alpha(E_0/\eta_0)e^{-\alpha x}}\neq s\frac{\partial}{\partial x}\bar{E}(x,s). \tag{17}$$

This time, the replacements for $E(x)$ and $E'(x)$ are not consistent. As a result, the correspondence principle fails in the inseparable case.

3.3 Tensile Loading on a Maxwell Bar With Displacement Boundary Conditions. A bar, made of Maxwell material, is loaded in tension as shown in Fig. 1(*b*). The lateral surface of the bar is traction free—so that the only nonzero stress is $\sigma_{xx}(x,t)\equiv\sigma(x,t)$. The boundary and initial conditions on the axial displacement $u(x,t)$ are

$$u(0,t)=0,\quad u(L,t)=v_0t;\quad u(x,0)=\varepsilon(x,0)=\sigma(x,0)=0 \tag{18}$$

where v_0 is a constant.

Elastic Solution. Using the usual equations (here $\varepsilon(x,t)$ is the axial strain)

$$\varepsilon(x,t)=\frac{\partial u(x,t)}{\partial x},$$
$$\sigma(x,t)=E(x)\varepsilon(x,t)=E_0e^{-\alpha x}\frac{\partial u(x,t)}{\partial x},\quad \frac{\partial\sigma(x,t)}{\partial x}=0 \tag{19}$$

together with the boundary and initial conditions (18), one gets the solution

$$u(x,t)=v_0t\left[\frac{e^{\alpha x}-1}{e^{\alpha L}-1}\right],\quad \varepsilon(x,t)=\frac{\alpha v_0te^{\alpha x}}{e^{\alpha L}-1},\quad \sigma(x,t)=\frac{\alpha E_0v_0t}{e^{\alpha L}-1}. \tag{20}$$

Note that since the stress must satisfy the equilibrium equation $\partial\sigma/\partial x(x,t)=0$, it must be independent of *x*.

Case (a) "Separable Class"—Viscoelastic Solution. The viscoelastic solution for this case is obtained easily by applying the correspondence principle. Carrying out the replacement

$$E(x)=E_0e^{-\alpha x}\Rightarrow s\bar{E}(x,s)=\frac{sE(x)}{s+E_0/\eta_0}=\frac{sE_0e^{-\alpha x}}{s+E_0/\eta_0},$$

one gets

$$\bar{\sigma}(x,s)=\frac{\alpha E_0v_0}{s[s+E_0/\eta_0][e^{\alpha L}-1]},$$
$$\sigma(x,t)=\frac{\alpha\eta_0v_0}{[e^{\alpha L}-1]}[1-\exp(-E_0t/\eta_0)]. \tag{21}$$

As expected from the correspondence principle, the solutions for $\varepsilon(x,t)$ and $u(x,t)$ can be easily shown to be the same as the elastic solutions $(20)_2$ and $(20)_1$.

Case (b) "Inseparable Class"—Viscoelastic Solution. It is easy to show that, in this case, an attempt to apply the correspondence principle fails. One gets a stress solution that is a function of *x*, and, therefore, does not satisfy equilibrium.

The boundary value problem to be solved is defined by the equations (see Fig. 1(*a*))

$$\frac{\partial\sigma}{\partial x}(x,t)=0,\quad \varepsilon(x,t)=\frac{\partial u}{\partial x}(x,t) \tag{22}$$

$$\frac{\partial\sigma}{\partial t}(x,t)+\frac{E(x)}{\eta(x)}\sigma(x,t)=E(x)\frac{\partial\varepsilon}{\partial t}(x,t) \tag{23}$$

together with $E(x)=E_0e^{-\alpha x}$, $\eta(x)=\eta_0$ and the boundary and initial conditions (18).

Taking the Laplace transform of (23), one gets

$$\bar{\sigma}(x,s)=\frac{sE(x)\bar{\varepsilon}(x,s)}{s+E(x)/\eta(x)}=s\bar{E}(x,s)\bar{\varepsilon}(x,s). \tag{24}$$

The stress must satisfy equilibrium $(22)_1$, i.e., it must be independent of *x*. Therefore, one can write

$$\bar{\sigma}(x,s)=sC(s),\quad \bar{\varepsilon}(x,s)=C(s)/\bar{E}(x,s) \tag{25}$$

where the function $C(s)$ must be obtained from boundary conditions.

Integrating $(25)_2$ with respect to *x*, and using the boundary conditions in (18), one has

$$\bar{u}(L,s)=\int_0^L\frac{C(s)}{\bar{E}(x,s)}dx=\frac{v_0}{s^2},\quad C(s)=\frac{v_0}{s^2I(s)} \tag{26}$$

where $I(s)$, with $E(x)=E_0e^{-\alpha x}$, $\eta(x)=\eta_0$, is

$$I(s)=\int_0^L \frac{dx}{E(x,s)}=\frac{s}{\alpha E_0}(e^{\alpha L}-1)+\frac{L}{\eta_0}. \quad (27)$$

From $(25)_1$, $(26)_2$, and (27), one obtains the Laplace transform of the stress, and then the stress as a function of x and t. The result is

$$\bar{\sigma}(x,s)=\frac{v_0}{s\left[\frac{s}{\alpha E_0}(e^{\alpha L}-1)+\frac{L}{\eta_0}\right]}, \quad \sigma(x,t)=\frac{\eta_0 v_0}{L}[1-e^{-bt}] \quad (28)$$

where $b=\alpha L E_0/[\eta_0(e^{\alpha L}-1)]$. With $\sigma(x,t)$ determined, $\varepsilon(x,t)$ is obtained directly from the viscoelastic constitutive Eq. (23). Integrating the resulting expression for $\partial\varepsilon/\partial t(x,t)$ with respect to t, and using the quiescent initial condition $\varepsilon(x,0)=0$, one gets the solution for the strain distribution in the bar. This is

$$\varepsilon(x,t)=\frac{\eta_0 v_0}{L E_0}\left[e^{\alpha x}-\frac{(e^{\alpha L}-1)}{\alpha L}\right][1-e^{-bt}]+\frac{v_0 t}{L}. \quad (29)$$

Note that, in this example, with the elastic strain distribution $\varepsilon(x,t)$ in $(20)_2$ independent of E, $\bar{\varepsilon}(x,s)$ from Eq. (24) *is not* the Laplace transform of the elastic solution $\varepsilon(x,t)$. In other words, contrary to the dictates of the correspondence principle, $\varepsilon(x,t)$ in Eq. (29) is different from the elastic strain solution in Eq. $(20)_2$.

Finally, integrating (29) with respect to x, and using the boundary conditions from (18), leads to the solution for the displacement field in the bar. This is

$$u(x,t)=\frac{\eta_0 v_0}{\alpha L E_0}\left[e^{\alpha x}+\frac{x}{L}(1-e^{\alpha L})-1\right][1-e^{-bt}]+\frac{v_0 x t}{L}. \quad (30)$$

3.4 Tensile Loading on a Maxwell Bar With Mixed Boundary Conditions. That the situation is quite subtle is evident from considering the same example as in Fig. 1(b), but this time with the boundary and initial conditions

$$u(0,t)=0, \quad \sigma(L,t)=\sigma_0 t; \quad u(x,0)=\varepsilon(x,0)=\sigma(x,0)=0 \quad (31)$$

where σ_0 is a constant.

Elastic Solution. The elastic solution for this problem is easily obtained. It is

$$\sigma(x,t)=\sigma_0 t, \quad \varepsilon(x,t)=\frac{\sigma_0 t e^{\alpha x}}{E_0}, \quad u(x,t)=\frac{\sigma_0 t e^{\alpha x}}{\alpha E_0}. \quad (32)$$

Case (b) "Inseparable Class"—Viscoelastic Solution. The strategy here is to try the correspondence principle first and then examine the resulting solutions. Applying the correspondence principle, one has

$$\sigma(x,t)=\sigma_0 t, \quad \bar{\varepsilon}(x,s)=\frac{\sigma_0 e^{\alpha x}}{s^2 E_0}+\frac{\sigma_0}{\eta_0 s^3},$$
$$\varepsilon(x,t)=\frac{\sigma_0 t e^{\alpha x}}{E_0}+\frac{\sigma_0 t^2}{2\eta_0}. \quad (33)$$

It is easy to show that these solutions satisfy the equilibrium Eqs. $(22)_1$ and the constitutive Eq. (23), together with the boundary and initial conditions (31). Therefore, they are correct. However, the correct solution for the displacement field $u(x,t)$, obtained by integrating $(33)_3$, is

$$u(x,t)=\frac{\sigma_0 t e^{\alpha x}}{\alpha E_0}+\frac{\sigma_0 t^2 x}{2\eta_0} \quad (34)$$

whereas the correspondence principle delivers the *wrong solution* $u(x,t)=\varepsilon(x,t)/\alpha$.

4 Concluding Remarks

In a recent paper in this journal, Paulino and Jin [1] have proved that the viscoelastic correspondence principle is valid for a class of functionally graded materials (FGMs) with separable relaxation functions. The present paper revisits this issue and examines the reasons behind the success or failure of the correspondence principle for viscoelastic FGMs. While material nonlinearities, moving boundaries, or moving loads (for example) are well-known reasons for the failure of the viscoelastic correspondence principle, to the best of the authors' knowledge, the reasons for the failure of the principle due to continuously spatially varying material (elastic and viscoelastic) properties have not been discussed before in the literature. Schapery [22] has considered this class of problems, but not for the usual situation in which the elastic material properties depend only on spatial coordinates. Also, it is not clear to the authors of the present paper whether anyone has noticed before that for the inseparable class of nonhomogeneous materials, the viscoelastic correspondence principle fails because of an inconsistency between the replacements of the moduli and of their derivatives (see Eqs. (16)–(17)).

As stated before, the correspondence principle always works for the "separable class" of materials, and does not, in general, for the "inseparable class" of viscoelastic materials. Examples of applications of the correspondence principle to FGM problems in the separable class are available in Refs. [5–7].

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References

[1] Paulino, G. H., and Jin, Z.-H., 2001, "Correspondence Principle in Viscoelastic Functionally Graded Materials," ASME J. Appl. Mech., **68**, pp. 129–132.
[2] Reiter, T., Dvorak, G. J., and Tvergaard, V., 1997, "Micromechanical Models for Graded Composite Materials," J. Mech. Phys. Solids, **45**, pp. 1281–1302.
[3] Noda, N., 1999, "Thermal Stresses in Functionally Graded Materials," J. Therm. Stresses, **22**, pp. 477–512.
[4] Erdogan, F., 1995, "Fracture Mechanics of Functionally Graded Materials," Composites Eng., **5**, pp. 753–770.
[5] Paulino, G. H., and Jin, Z.-H., 2001, "Viscoelastic Functionally Graded Materials Subjected to Antiplane Shear Fracture," ASME J. Appl. Mech., **68**, pp. 284–293.
[6] Paulino, G. H., and Jin, Z.-H., 2001, "A Crack in a Viscoelastic Functionally Graded Material Layer Embedded Between Two Dissimilar Homogeneous Viscoelastic Layers—Antiplane Shear Analysis," Int. J. Fract., **68**, 283–303.
[7] Jin, Z.-H., and Paulino, G. H., 2002, "A Viscoelastic Functionally Graded Strip Containing a Crack Subjected to In-Plane Loading," Eng. Fract. Mech., **69**, pp. 1769–1790.
[8] Yang, Y. Y., 2000, "Time-Dependent Stress Analysis in Functionally Graded Materials," Int. J. Solids Struct., **37**, pp. 7593–7608.
[9] Paulino, G. H., Chan, Y. S., and Fannjiang, A. C., 2002, "Gradient Elasticity Theory for Mode III Fracture in Functionally Graded Materials—Part I. Crack Perpendicular to the Material Gradation," ASME J. Appl. Mech., in press.
[10] Reddy, J. N., 2000, "Analysis of Functionally Graded Plates," Int. J. Numer. Methods Eng., **47**, pp. 663–684.
[11] Aboudi, J., Pindera, M. J., and Arnold, S. M., 1997, "Microstructural Optimization of Functionally Graded Composites Subjected to a Thermal Gradient via the Coupled Higher-Order Theory," Composites, Part B, **28**, pp. 93–108.
[12] Hirai, T., 1996, "Functionally Gradient Materials," Mater. Sci. Technol., Vol. **17B:** *Processing of Ceramics, Part 2*, R. J. Brook, ed., VCH Verlagsgesellschaft mbH, Weinheim, Germany, pp. 292–341.
[13] Paulino, G. H., Jin, Z.-H., and Dodds, R. H., 2002, "Failure of Functionally Graded Materials," *Comprehensive Structural Integrity*, Vol. 2, B. Karihaloo and W. G. Knauss, eds., Elsevier, New York, Chapter 13.
[14] Suresh, S., and Mortensen, A., 1998, *Fundamentals of Functionally Graded Materials*, Institute of Materials, London.
[15] Alex, R., and Schovanec, L., 1996, "An Anti-Plane Crack in a Nonhomogeneous Viscoelastic Body," Eng. Fract. Mech., **55**, pp. 727–735.
[16] Herrmann, J. M., and Schovanec, L., 1990, "Quasi-Static Mode III Fracture in a Nonhomogeneous Viscoelastic Body," Acta Mech., **85**, pp. 235–249.
[17] Herrmann, J. M., and Schovanec, L., 1994, "Dynamic Steady-State Mode III

Fracture in a Non-Homogeneous Viscoelastic Body,'' Acta Mech., **106**, pp. 41–54.
[18] Schovanec, L., and Walton, J. R., 1987, ''The Quasi-Static Propagation of a Plane Strain Crack in a Power-Law Inhomogeneous Linearly Viscoelastic Body,'' Acta Mech., **67**, pp. 61–77.
[19] Schovanec, L., and Walton, J. R., 1987, ''The Energy Release Rate for a Quasi-Static Mode I Crack in a Nonhomogeneous Linear Viscoelastic Body,'' Eng. Fract. Mech., **28**, pp. 445–454.
[20] Hilton, H. H., and Clements, J. R., 1964, ''Formulation and Evaluation of Approximate Analogies for Transient Temperature Dependent Linear Visco-Elastic Media,'' *Proceedings of the Conference on Thermal Loading and Creep*, Paper 12, pp. 6-17–6-24.
[21] Hashin, Z., 1965, ''Viscoelastic Behavior of Heterogeneous Media,'' ASME J. Appl. Mech., **32**, pp. 630–636.
[22] Schapery, R. A., 1978, ''A Method for Predicting Crack Growth in Nonhomogeneous Viscoelastic Media,'' Int. J. Fract., **14**, pp. 293–309.
[23] Christensen, R. M., 1971, *Theory of Viscoelasticity*, Academic Press, New York.