1 Equilibrium Equations of the von Mises Truss

We use the principle of virtual work to obtain the analytical solution of the von Mises truss.

$$\delta W_{\text{int}} = \delta W_{\text{ext}} \tag{1}$$

$$2AL\sigma\delta\varepsilon + C\left(u_1 - u_2\right)\left(\delta u_1 - \delta u_2\right) = P\delta u_1 \tag{2}$$

The strain measure chosen for these examples is the rotated engineering strain

$$\varepsilon = \frac{L_{\text{deformed}} - L}{L} \tag{3}$$

where

$$L_{\text{deformed}} = \sqrt{L^2 - 2Hu_2 + u_2^2} \tag{4}$$

and

$$\delta\varepsilon = \frac{u_2 - H}{L\sqrt{L^2 - 2Hu_2 + u_2^2}}\delta u_2 \tag{5}$$

Then the equilibrium equations are

$$2A\sigma\frac{u_2 - H}{\sqrt{L^2 - 2Hu_2 + u_2^2}} - C\left(u_1 - u_2\right) = 0 \tag{6}$$

and

$$C\left(u_1 - u_2\right) = P \tag{7}$$

To trace the equilibrium paths, P vs. u_1 and P vs. u_2, these curves can be parametrized by u_2. The equilibrium equations are combined, which yields

$$P = 2A\sigma\frac{u_2 - H}{\sqrt{L^2 - 2Hu_2 + u_2^2}} \tag{8}$$

and

$$u_1 = \frac{P}{C} + u_2 \tag{9}$$

2 Elastic Case of the von Mises Truss

For the elastic case of the von Mises truss, the equilibrium equation (Eqn. 8) is obtained using the relation

$$\sigma = E\varepsilon \tag{10}$$

2.1 Critical Spring Stiffness of the von Mises Truss

The stiffness of the spring in the von Mises truss dictates the overall behavior of the system. Stiffness below a critical value results in snap back behavior in the u_1 component, while values above do not. The critical case is realized when the tangent line to the P vs. u_1 is vertical (i.e. $\partial u_1/\partial P = 0$), and when $u_1 = u_2 = H$, and $P = 0$. Then the expression for the critical value of the spring stiffness is

$$C_{\text{cr}} = 2\frac{EA}{L}\left(\frac{L}{\sqrt{L^2 - H^2}} - 1\right) \tag{11}$$

3 Elastic-Plastic Case of the von Mises Truss

The analytical solution of the von Mises truss is based on the equilibrium equations derived in Section 1. The stages of the solution and limiting/critical values of the flow stress material property are derived and discussed in the following subsections.

3.1 Stages of the Elasto-Plastic Solution

The solution to the elasto-plastic von Mises truss is comprised of two elastic stages and two plastic stages. A subscript in parentheses references the stage.

Stage 1: Elastic Compressive Stage

The first stage is equal to the elastic solution, therefore the $\sigma = E\varepsilon$. Stage 1 starts at the undeformed configuration (i.e. $u_1 = u_2 = 0$) and stops when the flow stress is reached in compression (i.e. when $\sigma = -\sigma_Y$). The limit on u_2 for stage 1 must be less than H, and is

$$u_{2\,\max}^{(1)} = H\left[1 - \sqrt{1 - \left(\frac{L}{H}\right)^2 \left(\frac{\sigma_Y}{E}\right)\left(2 - \frac{\sigma_Y}{E}\right)}\right] \tag{12}$$

Stage 2: Plastic Compressive Stage

The second stage is plastic and the bars are in compression, therefore $\sigma = -\sigma_Y$. Stage 2 continues from the last point in stage 1 until the structure is flat, i.e.

$$u_{2\,\max}^{(2)} = H \tag{13}$$

Stage 3: Elastic unloading stage

The third stage is again elastic, and the stress is $\sigma = E\left(\varepsilon - \varepsilon_{\min}\right) - \sigma_Y$, where $\varepsilon_{\min}$ is strain at the end of stage 2, i.e. when $u_2 = H$

$$\varepsilon_{\min} = \sqrt{1 - \left(\frac{H}{L}\right)^2} - 1 \tag{14}$$

Stage 3 continues from the last point in stage 2 until the flow stress is reached again, this time in tension. The limit on u_2 for stage 3 is found by equating $\sigma = E\left(\varepsilon - \varepsilon_{\min}\right) - \sigma_Y$ with $\sigma = \sigma_Y$, which results in

$$u_{2\,\max}^{(3)} = H + 2\sqrt{\frac{L^2\sigma_Y\left[\frac{\sigma_Y}{E} + \sqrt{1 - \left(\frac{H}{L}\right)^2}\right]}{E}} \tag{15}$$

Stage 4: Plastic Tensile Stage

The fourth stage is plastic, but now the bars are in tension, so $\sigma = \sigma_Y$. In this final stage there is a horizontal asymptote at $P_{\max} = 2A\sigma_Y$.

3.2 Limits on the Flow Stress

The flow stresses of the elasto-plastic von Mises truss are selected such that the material will reach the the flow stress while the bars are in compression (i.e. before the structure reaches the plane configuration). Hence, the flow stress must be less than the maximum stress of the elastic stage, which occurs when $u_2 = H$.

$$\sigma_{Y,\max} = E\left[1 - \sqrt{1 - \left(\frac{H}{L}\right)^2}\right] \tag{16}$$

Additionally, depending on the value of the flow stress, the first plastic stage will start before or after the load limit point of the elastic solution, respectively. Therefore, the critical flow stress is equal to the stress at the load limit point

$$\sigma_{\mathrm{cr}} = E\left\{1 - \frac{\left[\left(L^2 - H^2\right)L\right]^{1/3}}{L}\right\} \tag{17}$$