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# Pointwise error estimates for boundary element calculations

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## ABSTRACT

An essential ingredient for all adaptive boundary integral methods is a reliable estimate of the local error. This paper investigates an *a posteriori* error indicator based upon the evaluation of hypersingular equations. Computational experiments were performed for the two dimensional Laplace equation on interior and exterior domains, employing Dirichlet, Neumann, and mixed boundary conditions. The results indicate that the error indicator successfully tracks the form of the exact error curve. Moreover, a reasonable estimate of the magnitude of the error was also obtained.

## INTRODUCTION

A wide variety of adaptive Boundary Element Methods have been developed, including grid refinement ($h$-method),[1-5] higher order polynomial interpolation ($p$-method),[6,7] grid redistribution ($r$-method),[8,9] and combinations of these techniques.[10,11] The recent book edited by Brebbia and Aliabadi[12] provides a good overview of this area, together with a more complete list of references. An essential requirement for all adaptive techniques is a reliable and computation-

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ally feasible procedure for estimating the local error. For collocation approximations, many different local indicators of error have been employed,[13] most of which involve perturbing the original calculation in some manner. While these methods have intuitive appeal, very little justification, either by means of theoretical analysis or experimental testing, has been developed. Moreover, many of these indicators rely on knowledge of the boundary conditions everywhere on the boundary, not just the nodal points. In many applications, such as time-dependent moving boundary problems, this information is not available.

The purpose of this paper is to introduce and evaluate, by means of computational experiments, a local error estimate based upon hypersingular equations. These equations have proven to be highly useful in a variety of situations, and their definition and numerical evaluation are now well understood (see, for example, Krishnasamy *et al.*[14] or Gray[15]). A distinct advantage of this approach is that the indicator does not require boundary values away from the nodal points and it does not involve any adjustable parameters. Moreover, the results of the computational experiments indicate that it provides a very reliable tracking of the exact error.

An error indicator which relies upon the ability to evaluate hypersingular integrals was recently proposed by Ingber and Mitra.[16, 17] The approach suggested in the present paper is therefore related to that discussed by these authors, but there is a fundamental difference in the two techniques. As will be discussed further below, the idea motivating the Ingber and Mitra procedure is to measure the discrepancy between the imposed boundary conditions and the boundary element solution. For the indicator proposed herein, the measure of error is taken to be the amount by which the solution to the standard boundary integral equation fails to satisfy the (equally valid) hypersingular equation.

## LOCAL ERROR ESTIMATE

For simplicity, the error indicator will be defined in the context of the two-dimensional Laplace equation $\nabla^2\phi = 0$. However, the technique should be applicable to any boundary integral formulation.

The boundary integral formulation for this equation can be written as

$$\phi(P) + \int_\Gamma \phi(Q)\frac{\partial}{\partial n}G(P,Q)\,d\Gamma(Q) = \int_\Gamma \phi_n(Q)G(P,Q)\,d\Gamma(Q) \ , \qquad (1)$$

where $\Gamma$ is the boundary of the domain $\mathcal{D}$, $\phi$ is the potential, $\phi_n$ is the boundary flux (normal derivative of $\phi$), and the point source potential

$$G(P,Q) = -\frac{1}{2\pi}\log\|Q-P\| \qquad (2)$$

will be employed for the Green's function. Equation (1) is valid for points $P$ inside the domain $\mathcal{D}$ (thus the coefficient 1 for the leading term). However, as discussed by Lutz and Gray,[18] it is also valid for $P$ on the boundary, provided the singular integrals are defined as a limit to the boundary. Moreover, for $P \in \mathcal{D}$, the integrands are not singular, and the derivative of Eq. (1) with respect to $P$ can be computed by interchanging the order of differentiation and integration. The resulting boundary integral equation for the gradient of the potential can therefore be expressed as

$$(\nabla\phi \cdot \mathbf{D})(P) + \int_\Gamma \phi(Q)\nabla\frac{\partial}{\partial n}G(P,Q)\cdot\mathbf{D}\,d\Gamma(Q) = \int_\Gamma \phi_n(Q)\nabla G(P,Q)\cdot\mathbf{D}\,d\Gamma(Q)\ , \quad (3)$$

where $\mathbf{D} = (D_1, D_2)$ is any specified direction vector, $||\mathbf{D}|| = 1$. Once again, a well defined (hypersingular) boundary equation results by defining the singular integrals as a limit in which $P$ approaches $\Gamma$.[15,19]

Assume now that a particular problem has been solved using Eq. (1), resulting in approximate solutions $\phi_A$ and $(\phi_n)_A$ for the unknown boundary values of potential and flux. These approximate solutions, together with the specified boundary conditions, determine these boundary functions at all the nodal points employed to discretize the geometry. Although these functions have been determined to satisfy Eq. (1) (in whatever sense this equation was approximated), these values are not necessarily consistent with Eq. (3).

The proposed error indicator $\mathcal{E}(P)$ is defined as the error which arises when the approximate solution is substituted into Eq. (3), with $\mathbf{D} = \mathbf{N}(P)$, the normal $P$. Thus,

$$\mathcal{E}(P) = (\phi_{\mathbf{N}})_A(P) + \int_\Gamma \phi_A(Q)\frac{\partial}{\partial N}\frac{\partial}{\partial n}G\,d\Gamma - \int_\Gamma (\phi_n)_A(Q)\frac{\partial}{\partial N}G\,d\Gamma\ . \quad (4)$$

Note that $n$ and $N$ refer to a field point $(Q)$ normal and a source point $(P)$ normal, respectively. As all quantities on the right hand side of Eq. (4) are known, $\mathcal{E}(P)$ can be calculated. Computational tests were performed to compare the exact error with the values predicted by Eq. (4). The next section presents the results of some of these tests, with a more complete discussion being reported elsewhere.

As discussed above, an error indicator utilizing the hypersingular equation was first proposed by Ingber and Mitra.[16] They define the indicator for a boundary element $\Gamma_j$ as

$$\begin{array}{ll} \left\{\int_{\Gamma_j}[\phi-\phi_A]^2\,d\Gamma\right\}^{\frac{1}{2}} & \Gamma_j \in \Gamma_d \\ \left\{\int_{\Gamma_j}[\phi_n-(\phi_n)_A]^2\,d\Gamma\right\}^{\frac{1}{2}} & \Gamma_j \in \Gamma_n\ , \end{array} \quad (5)$$

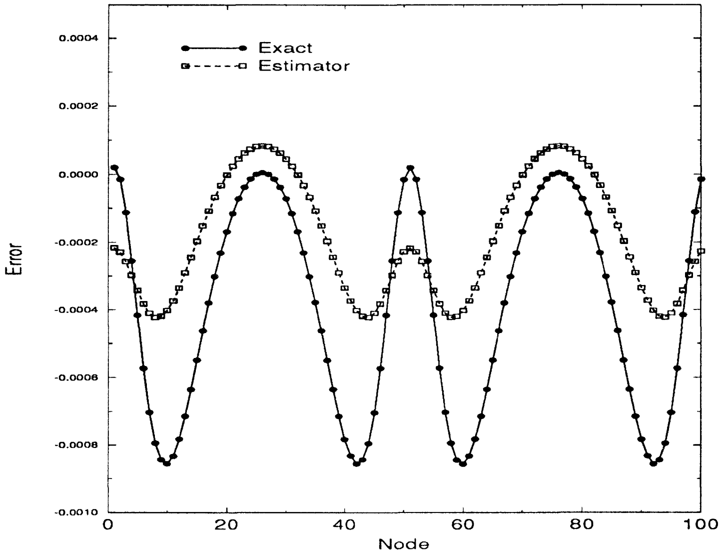

Figure 1: $\mathcal{E}(P)$ vs. the exact error for the elliptical geometry $a^2 = 1$, $b^2 = 0.8$.

where $\Gamma_d$ and $\Gamma_n$ are the parts of the boundary having respectively Dirichlet and Neumann boundary conditions. Thus, computing this indicator (and others) requires knowledge of $\phi$ and $\phi_n$ everywhere on the (approximate) boundary $\Gamma_j$. For some types of calculations, this data will not be known. On the other hand, Eq. (5) clearly requires less computation than Eq. (4).

## CALCULATIONS

The computational experiments reported herein examine the performance of the proposed error indicator (Eq. (4)) for interior Dirichlet and mixed boundary value problems. For the Dirichlet problem, the prescribed boundary values were specified as

$$\phi(x,y) = x^2 - y^2 \ . \tag{6}$$

Two elliptical geometries, $x^2/a^2 + y^2/b^2 = 1$, were tested, and the values of $\mathcal{E}(P)$ at the nodes, along with the exact errors, are shown in Figs. 1 ($a^2 = 1$, $b^2 = 0.8$) and 2 ($a^2 = 1$, $b^2 = 0.5$). The exact error is defined as the difference between the exact and numerical (Eq. (1)) solutions. The last ellipse (Fig. 2) was also employed for the mixed boundary value problem. In this case, $\phi$ as in Eq. (6) was specified on the top of the ellipse, while $\phi_n$ (obtained by differentiating Eq. (6)) was given on the other half of the boundary. The results

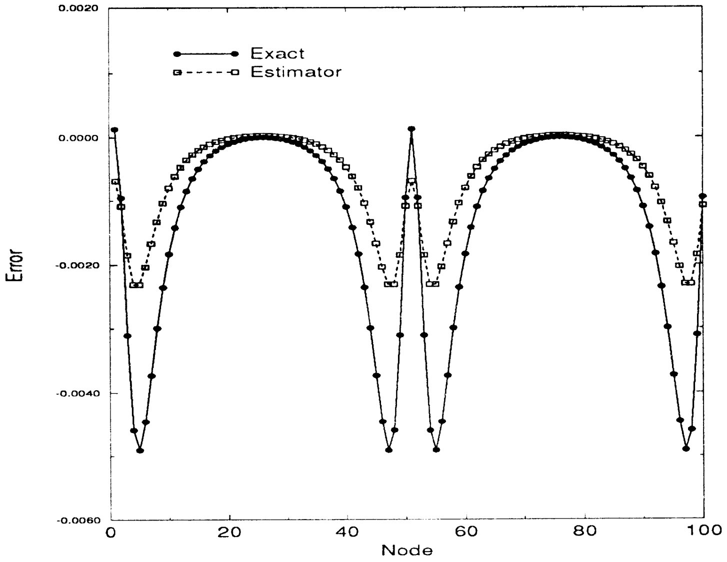

Figure 2: $\mathcal{E}(P)$ vs. the exact error for the elliptical geometry $a^2 = 1$, $b^2 = 0.5$.

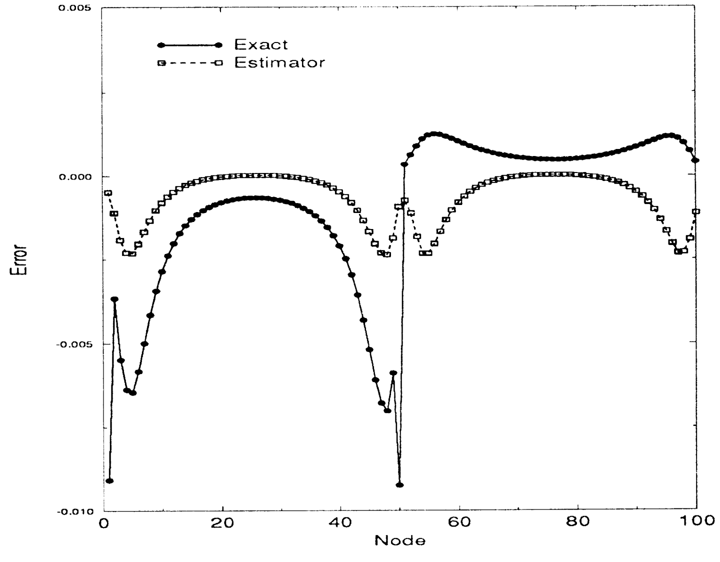

Figure 3: $\mathcal{E}(P)$ vs. the exact error for the mixed boundary value problem, elliptical geometry $a^2 = 1$, $b^2 = 0.5$.

of this calculation are shown in Fig. 3.

In all cases, $\mathcal{E}(P)$ does an excellent job predicting the shape of the error curve. The regions of the boundary where the approximations should be improved are accurately identified, and thus reliable information would be passed to an adaptive mesh refinement procedure. Moreover, $\mathcal{E}(P)$ provides a fairly reasonable estimate of the magnitude of the error.

## CONCLUSIONS

Computational experiments indicate that substituting the approximate boundary element solution into the corresponding hypersingular integral equation provides a remarkably faithful tracking of the exact error curve. The primary disadvantage of this technique is computational cost: an integration over the boundary is required to evaluate $\mathcal{E}(P)$ for every $P$. Thus, the possibility of reducing this cost by neglecting 'far-field' integrals should be investigated. An advantage of this technique is that it allows a pointwise estimation of the discretization error. In many design problems, one is only interested in assessing the error at particular critical regions. Therefore, the error $\mathcal{E}(P)$ could be evaluated just at points in these specific regions.

Another subject for future research is the extension of this technique to crack geometries. On a crack surface, the hypersingular equation (together with the standard equation) is already employed in the solution of the problem,[19] and thus there is no longer a 'free' equation available for error estimation. One possibility is to use the hypersingular equation, the directional derivative being taken tangent to the crack surface.

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